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Relativity of rotation

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Abstract. A convention for light velocity as a function of the coordinates fixes the geometry of space–time; alternatively, a convention for geometry fixes light velocity. The latter type of convention permits the use of flat space–time, but global distances and time are given by non-holonomic expressions, which amounts to measuring such distances and times in terms of coordinate-dependent units. The relativity of rotation is discussed with the convention of flat space–time. Effects examined include the Thomas precession, the Ehrenfest paradox, the curvature of light rays relative to rotating universes (isolated systems) but not with respect to rotating local systems, the clock paradox for circular motion, the time difference measured in the Hafele–Keating experiment, the Sagnac effect (particularly how it disappears when a local rotating system is expanded to become a rotating universe), and the magnetic-type gravitational fields that arise in rotating frames of reference (for the convention of flat space–time). The necessity to distinguish between rotation of a local system (a disc) and rotation of a perfectly isolated system (a universe) is emphasized.

1. Introduction

Adopting the convention of flat space–time, which is always a valid option (Browne 1976), it becomes necessary to consider not merely the motion of matter with respect to some chosen reference system, but also the motion of aether fluid. The latter motion will be non-holonomic, implying non-integrable radar distances which have to be interpreted in terms of the coordinate-dependent units first considered by Dicke (1962). Aspects of the relativity of rotation will be considered within this framework.

Problems connected with the relativity of rotation still arouse considerable controversy, as shown by communications following remarks by Atwater (1970). Atwater drew attention to two problems. Firstly, would a scale on the circumference of a relativistically rotating transparent disc appear contracted relative to a similar scale on an underlying stationary disc when a light flash above the discs is used to obtain an instantaneous photograph of superimposed markings? Fitzgerald contraction of the circumference would imply asymmetry without proportional contraction of the radius (the Ehrenfest paradox); contraction of elements of the radius might be attributed to their accelerations.

Secondly, the electromagnetic equations in a rotating frame of reference, when obtained from those in an inertial frame by a Galilean transformation of coordinates, ought not, in Atwater's view, to be correct. General covariance permits any transformation, but it does not demand that the velocity of light remain constant as happens after the Lorentz transformation. A holonomic coordinate transformation does not alter the geometry of space–time, which must therefore remain flat. But if the angular velocity is corrected by addition of the radius-dependent Thomas precession the transformation will be non-holonomic.

The Thomas precession became a major consideration in the ensuing communications (Marsh 1971, Weinstein 1971, Whitmire 1972a, b). Due to the Thomas precession angular velocity becomes a function of radius $\omega(r)$, and questions discussed were how to reconcile this differential rotation with the elastic properties of specific media and whether the differential rotation can be directly measured. The relativistic mechanics of a rotating ring were examined by McCrea (1971) on the basis of the definition of rigidity given by Hogarth and McCrea (1952). Some interesting remarks on the rotation of rigid bodies in relativity theory have been made by Rosen (1947).

The curvature of light rays relative to rotating frames of reference has been considered by Jennison and co-authors (Jennison 1963, Jennison 1964, Davies and Jennison 1975, Ashworth and Jennison 1976). One's first reaction to this topic is curvature with respect to what? With the choice of Einstein's space–time one curvilinear reference system is as good as another, so that implicitly the curvature is measured with respect to a Cartesian reference system in flat space–time. The choice of the latter reference system requires validation (Browne 1976).

A fundamental consideration, to which only Post (1967) seems to have given recognition, is the necessity to distinguish between a rotating local system and a rotating universe, where a 'universe' is defined as the smallest perfectly isolated system consistent with Mach's principle. A transformation of coordinates suffices to describe effects for the latter case, but not for the former case which is physically different. In other words, whether or not background matter rotates is physically significant.

For bibliography up to 1966 see Arzelies (1966).

2. Arbitrariness of geometry and aether motion

Milne (1935) has emphasized that the only observations open to an observer at some world point ('particle-observer') are those which can be made with a natural clock and a goniometer (for measurement of the directions of outgoing and incoming light signals). The observer, in effect, finds himself in the situation of the astronomer. It is an illusion to think that rigid scale lengths or constant rate clocks can be transported from world point A to world point B, because ultimately the rigidity of the scales or the constancy of the clock rates can be determined only by radar-type experiments.

The profundity of Milne's work, and its implications, have not been adequately appreciated. Inevitably a description of the natural world is based on an arbitrary convention. The convention may be one of two types; either the geometry of space–time is chosen arbitrarily, in which case the optical properties of the aether medium are determined, or the optical properties of the aether medium are chosen arbitrarily, in which case the geometry of space–time is determined. Classical physics is based on a convention of the former type, whilst Einstein's general theory of relativity adopts a convention of the latter type. The simplest choice for geometry is the Euclidean geometry which obtains on flat space–time. Among the reference systems that are possible on flat space–time an inertial system I has special significance. The simplest choice for light propagation is to assume that the velocity of light is independent of direction and has the constant value c independently of the world point considered. In fact this is true only for a special reference system $\hat{\mathrm{R}}$, and of course $\hat{\mathrm{R}}$ belongs to a space–time curved in a very special manner.

Einstein proposed that the laws of physics should be covariant for transformations from $\hat{R}$ to any other reference system R belonging to the same space–time. This permits general holonomic transformations of space and time coordinates, which are of the form:

$$dx^\alpha = X^\alpha{}_{,\mu}\, d\hat{x}^\mu \tag{1}$$

where a comma denotes partial differentiation with respect to $\hat{x}^\mu$.

The arbitrariness of geometry is expressed by extending covariance to non-holonomic transformations of the coordinates of the type

$$dx^{*\alpha} = X^{*\alpha}_{\mu}\, dx^\mu. \tag{2}$$

Such a transformation introduces a reference system belonging to a different space–time. In particular, $x^{*\alpha}$ might be Cartesian coordinates in flat space–time. The price paid for such a simplification is that global distances and times are given by non-integrable expressions, and must therefore be interpreted in terms of coordinate-dependent units for distance and time as originally suggested by Dicke (1962).

The optical properties of the aether medium may be described either by assigning to the aether a refractive index tensor, or else, by giving the aether fluid a velocity field which may vary with time. In four dimensions the spatial reference points of a curvilinear system of coordinates (i.e. the points x^i = constants, where $i = 1, 2, 3$) move with respect to each other like particles of a fluid. The special significance of the reference system $\hat{R}$ is that the spatial reference points are at rest relative to the aether fluid. It is not surprising, therefore, that aether effects disappear from Einstein's theory. With respect to any other reference system R belonging to the same space–time the aether fluid will have holonomic motion and the velocity of light will no longer be constant. However, this is not a new physical situation. With respect to a reference system R* belonging to a different space–time, the aether fluid will have non-holonomic motion, and this will determine the coordinate dependence of the units.

Not only is $\hat{R}$ a reference system at rest relative to the aether fluid, but it is also a reference system such that the gravitational field everywhere vanishes. With respect to any reference system other than I particles of matter have fictitious accelerations, and a compensating gravitational field is necessary to remove the inertial forces associated with the fictitious accelerations. Thus, attached to each non-inertial reference system is a compensating gravitational field. The idea behind Einstein's theory of gravitation is to achieve a reference system whose compensating gravitational field everywhere annuls the permanent gravitational field of the matter distribution. This unique reference system is $\hat{R}$. With respect to any other reference system R belonging to the same space–time as $\hat{R}$ there will arise what Møller (1972) calls a 'non-permanent' gravitational field. With respect to a reference system R* belonging to a different space–time there will arise a permanent gravitational field, and when R* becomes I this will be a Newtonian-type field.

For gravitational field variables one is presented with several choices. One is the velocity field of the aether fluid (Browne 1976). Another would be the $X^{*\alpha}_{\mu}$ appropriate for transformation from $\hat{R}$ to R*; in this case, note that the addition to $X^{*\alpha}_{\mu}$ of $X^{*\alpha}_{,\mu}$ has no physical significance (since it does not alter the geometry of space–time), and indeed might be regarded as a generalization of the gauge transformation. For use of X^{α}_{μ} as field variables see for example Tupper (1971).

For a more detailed account of the above ideas see Browne (1976). They have been summarized here only in order to define the framework for discussion of relativistic effects associated with rotation.

3. The principles of Mach and of general covariance

When general curvilinear coordinates are introduced in four dimensions, the reference for motion can be visualized as an arbitrarily moving fluid, and indeed for the special system $\hat{R}$ this fictitious fluid can be identified with the aether fluid. A consequence is that the accelerations of material particles are changed in an apparently arbitrary manner depending on the choice of coordinate system. Actually the arbitrary contribution to the accelerations is annulled by a compensating gravitational field which is associated with each reference system. The source of this compensating field is matter of the universe in the large. The necessity to include the compensating field in a generally covariant description expresses that such a description is valid only for an inertially isolated system. Now according to Mach's principle nothing less than a universe is inertially isolated; a 'universe' will be regarded as a perfectly isolated system by definition, although it may have a finite radius—the Hubble radius. The distinction between global and local principles of relativity has been made by Treder (1970). What is clear is that the reconciliation of the principle of general covariance with the principle of Mach, of so much concern to Einstein, lies simply in a restriction on the system which is capable of covariant description—a restriction which seems not to apply for special relativity.

The principle of general covariance implies that physical situations which can be related to each other by a holonomic transformation of coordinates are physically indistinguishable. Let us divide our isolated system into a local system together with what might be described as 'background matter'. Consider, then, the following four situations: (*a*) stationary local system, stationary background matter; (*b*) rotating local system, rotating background matter; (*c*) rotating local system, stationary background matter; and (*d*) stationary local system, rotating background matter. Situations (*a*) and (*b*) can be related to each other by transformation to or from a rotating frame of reference, and hence they are physically indistiguishable. The same can be said for situations (*c*) or (*d*). But (*a*) or (*b*) are different physical situations to (*c*) or (*d*), and there are several phenomena by which they can be distinguished.

Post (1967), for example, enumerates the following three distinctions.

(i) 'The Barnett experiment: rotating the magnetizable bar or rotating the frame of reference instead of the bar are clearly different operations from a physical point of view. Rotation of the bar gives magnetization; rotation of the frame of reference does not.

(ii) The Oppenheimer paradox: rotating a charged spherical condenser or rotating the frame of reference instead of the condenser are physically different operations. Rotation of the condenser produces an external magnetic field; rotation of the frame of reference does not.

(iii) The rotational Fresnel–Fizeau experiment (medium rotating, mirrors stationary) and the Dufour–Prunier experiment (medium stationary, mirrors rotating) are physically not equivalent'.

There is ample evidence, therefore, that experiments performed on some local system are influenced by whether or not background matter happens to be rotating relative to one's frame of reference. An example is the Sagnac effect, which provides a direct measure of the angular velocity of the local system relative to the background matter. What happens if the Sagnac experiment is done on a rotating universe rather than a rotating local system? Obviously there can be no phase change, since it does not matter if we transform to an inertial frame. How this happens is discussed in § 9.

4. The Thomas precession

Let observer S, using axes X–Y, view observer S′ in uniform motion with velocity u along the positive X axis, and view also observer S″ in uniform motion with velocity v along the positive Y axis. Then, if the axes used by S′ and by S″ are parallel in the view of S, they cannot be parallel in the view of either S′ or S″. Thus, by the relativistic transformation of velocity S′ finds for the components of motion of S″ the values, $-u$, $v(1-u^2/c^2)^{1/2}$, 0, whereas S″ finds for the components of motion of S′ the values, $u(1-v^2/c^2)^{1/2}$, $-v$, 0. Thus S′ concludes that S″ moves with velocity w in a direction making angle θ' with his X' axis, where $\tan\theta' = -(1-u^2/c^2)^{1/2}(v/u)$, whilst S″ concludes that S′ moves with velocity $-w$ in a direction making angle θ'' with his X'' axis, where $\tan\theta'' = -(1-v^2/c^2)^{-1/2}(v/u)$. Since the direction of the relative velocity between S′ and S″ is a reference direction it follows that the axes of S″ and S are no longer parallel in the view of S′, and also that the axes of S′ and S are no longer parallel in the view of S″. The difference between θ' and θ'' will be the relative orientation of the axes of S′ and of S″, namely

$$\theta = \theta' - \theta'' = \tan^{-1}\left(\frac{v(1-u^2/c^2)^{1/2}}{u}\right) - \tan^{-1}\left(\frac{v}{u(1-v^2/c^2)^{1/2}}\right). \tag{3}$$

With regard to the magnitude of the relative velocity of S′ and S″ one finds

$$w^2 = u^2 + v^2 - u^2v^2/c^2. \tag{4}$$

Let us replace v by the infinitesimal dv, so that

$$d\theta = u^{-1}\, dv[(1-u^2/c^2)^{1/2} - 1]. \tag{5}$$

If $d\boldsymbol{v}$ is the velocity gained by acceleration $\boldsymbol{a}$ over time interval $\gamma\, dt$, then $d\boldsymbol{v} = \gamma u^{-1}\boldsymbol{u} \times \boldsymbol{a}\, dt$, in which case

$$\boldsymbol{\omega}_T = -\frac{d\theta}{dt} = -\frac{\boldsymbol{u}\times\boldsymbol{a}}{u^2}(1-\gamma) = \frac{\boldsymbol{u}\times\boldsymbol{a}}{c^2}\frac{\gamma}{1+\gamma} \tag{6}$$

where $\gamma = (1-u^2/c^2)^{-1/2}$ and $\boldsymbol{\omega}_T$ is the Thomas precession.

When the motion is circular, we may put $u = \omega r$ and $dv = \omega^2 r\, dt$, in which case

$$\boldsymbol{\omega}_T = (1-\gamma)\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_0 \tag{7}$$

where

$$\omega_0 = \frac{\omega}{(1-\omega^2r^2/c^2)^{1/2}} \qquad \omega = \frac{\omega_0}{(1+\omega_0^2r^2/c^2)^{1/2}}$$
$$\gamma = (1-\omega^2r^2/c^2)^{-1/2} = (1+\omega_0^2r^2/c^2)^{1/2}. \tag{8}$$

Thus, two angular velocities arise, ω_0 and ω. ω_0 is constant, but ω is a function of r due to the Thomas precession. Note that $\omega_0 r \to \infty$ as $r \to \infty$, but $\omega r \to c$ as $r \to \infty$.

The Thomas precession, as is clear from its derivation and more particularly from the derivation given by Furry (1955), is part of the composite picture that is built up by Lorentz transformations from local comoving reference systems. In this respect it is analogous to the variation of mass with velocity, and it arises in the same situations.

One way to visualize the Thomas precession is as follows. Consider a rotating disc, and introduce two frames of reference: (i) an inertial frame S with origin at the centre of rotation O; and (ii) a frame S′ whose origin O′ is fixed to the disc at radial distance r and which is freely pivoted about O′. Let observers S and S′ orientate their axes with respect to the direction of a distant star. The vector OO′ rotates with angular velocity ω with respect to the axes of S, but with ω_0 relative to the axes of S′. Yet both sets of axes point continually to a distant star. Clearly S and S′ must disagree about what constitutes a revolution of OO′. S monitors successive transits of O′ past a fixed marker, whilst S′ sees the marker plus the universe rotate about O; if O precesses at the same time about O′ then S and S′ can disagree about what constitutes a revolution of OO′.

Another derivation of the relation between $\omega(r)$ and ω_0 follows from generalization of the relation $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla}\times\boldsymbol{v}$ into covariant form, where $\boldsymbol{v}$ is the velocity field of a rotating system (Hill 1946, Rosen 1947). The generalization is

$$\omega_{\mu\nu} = \tfrac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}) = \tfrac{1}{2}(u_{\mu,\nu} - u_{\nu,\mu}). \tag{9}$$

Adopting an inertial frame of reference and Cartesian coordinates

$$\begin{aligned} u_1 &= -u^1 = \omega_0 y \qquad & u_2 &= -u^2 = -\omega_0 x \\ u_3 &= -u^3 = 0 \qquad & u_4 &= u^4 = (1+\omega_0^2 r^2/c^2)^{1/2}. \end{aligned} \tag{10}$$

The condition $u_\lambda u^\lambda = 1$ determines u_4. In terms of these coordinates

$$\omega_{12} = \omega_0 \qquad \omega_{14} = -\tfrac{1}{2}\gamma^{-1}\omega_0^2 x \qquad \omega_{24} = -\tfrac{1}{2}\gamma^{-1}\omega_0^2 y. \tag{11}$$

Since $u = v(1-v^2/c^2)^{-1/2}$, where $u = (u_1^2+u_2^2)^{1/2}$, we obtain

$$u = \omega_0 r \qquad v = \omega r \tag{12}$$

where ω and ω_0 are related as in (8).

An experimental test for the differential rotation $\omega(r)$ arising from the Thomas precession has been suggested by Weinstein (1971). The idea is simply to measure the angle between the directions of arrival of light signals at the centre of rotation O from sources fixed to a rotating disc at radii r_1 and r_2. If the differential rotation is not accompanied by appropriate contraction of the disc, this angle should change monotonically with time. The contraction would have to reduce each radius r to r' where $r' = r/\gamma$ and $dr' = dr/\gamma^3$. The distance of each source from O can be measured only by 'radar ranging'. Thus the contraction would imply that the Euclidean radii are less than the radar radii, presumably because light rays are curved. We shall see that light rays are indeed curved when background matter of the universe rotates, but not when the rotating system is local.

5. Curved light rays

Consider the path of a light ray relative to a rotating system which may be either (i) a rotating isolated system or universe, or (ii) a rotating local system or disc. In order to measure the path of the light signal, we may suppose that it passes through a sequence of slits which are fixed to the rotating system. The rotation is then stopped and the locus of slits examined by means of further light rays. Rotation of a universe can be stopped only by transforming to the co-rotating frame of reference, but rotation of the local disc can be stopped by means of an external couple. We consider a ray which passes through O (figure 1).

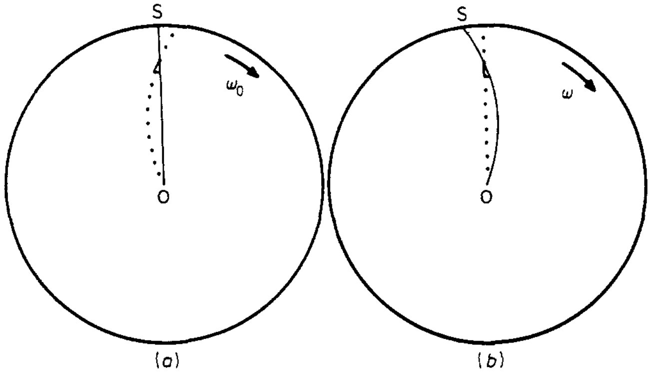

Figure 1. Locus of slits attached to a rotating system in positions so as to transmit a light ray to the centre of rotation O: (*a*) when background matter is stationary, and (*b*) when background matter co-rotates.

Let successive slits be assigned polar coordinates (r, ϕ) and $(r+\delta r, \phi+\delta\phi)$. In the case of the rotating universe the second slit moves transverse distance $r\,\delta\phi$ with velocity ωr whilst the light signal propagates arc distance δs with velocity c. We can therefore write

$$r\,\mathrm{d}\phi/\mathrm{d}s = \omega r/c. \tag{13}$$

On relating ω to ω_0 by (8) and noting that $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2$, the polar equation (13) becomes

$$r\,\mathrm{d}\phi/\mathrm{d}r = \omega_0 r/c \tag{14}$$

which integrates to $\omega_0 r/c = \phi - \phi_0$, where ϕ_0 is a constant. The locus of slits is therefore an Archimedes spiral. Because the rotating system is perfectly isolated, the principle of covariance can be used to infer that whatever positionings of the slits transmit the ray when the universe rotates must also transmit a ray when the universe is stationary; these two physical situations are in fact the same situation since they are related by a holonomic transformation of coordinates. The transformation, $\phi' = \phi - \omega_0 t$ and $r = ct$, changes the Archimedes spiral into $\phi' = \phi_0$, a straight line.

In the case of a local rotating system, the positions of the slits must be such as to transmit a ray when the system rotates and background matter is stationary, and also when the system is stationary and background matter rotates (see § 3). In fact the ray path is controlled primarily by the background matter, and therefore will be a straight line in the former situation and an Archimedes spiral in the latter situation. Clearly slits positioned in an Archimedes spiral relative to the local disc will transmit the ray for either situation. Note that this is a different physical situation to the rotating universe; relative to the rotating system the locus of slits now is an Archimedes spiral, whereas previously it was a straight line.

If we consider a ray that does not pass through the centre of rotation O, but has closest distance a to O, then for stationary system and stationary background we have

$$r \cos \phi = a \qquad r \sin \phi = ct. \tag{15}$$

The Galilean transformation, $\phi' = \phi - \omega_0 t$, leads to

$$\cos[\phi' - \omega_0 (r^2 - a^2)^{1/2}/c] = a/r. \tag{16}$$

For $a = 0$ this reduces to an Archimedes spiral. We would expect (15) to apply to stationary background (whether or not the local system rotates) and (16) to apply to rotating background (whether or not the local system rotates).

Jennison (1963) has obtained a circular ray path relative to a rotating system (he does not distinguish between local and isolated rotating systems). His argument is based on the angles of aberration for Lorentz observers at different radii, and his result is a circle rather than an Archimedes spiral because he fails to take account of the dependence of ω on r. Using the aberration formula, $\cos \theta' = (\cos \theta - \beta)/(1 - \beta \cos \theta)$ where $\beta = \omega r/c$, we see that a ray perpendicular to the motion for S′ ($\cos \theta' = 0$) will not be perpendicular to the motion for S ($\cos \theta = \beta$). Its direction differs from the normal to the motion by angle ϕ, where $\sin \phi = \beta$. The normal to the motion is just the radius vector, and hence the polar equation of the ray path is

$$r\, \mathrm{d}\phi/\mathrm{d}r = \tan \phi \qquad \sin \phi = \omega r/c \tag{17}$$

from which is obtained

$$\frac{r\, \mathrm{d}\phi}{\mathrm{d}r} = \frac{\omega r}{c(1 - \omega^2 r^2/c^2)^{1/2}} = \frac{\omega_0 r}{c}. \tag{18}$$

If ω were treated as constant (as does Jennison), integration of (18) yields $\omega r/c = \sin \phi$. The polar equation for a circle of radius R with origin on the circle is $r = 2R \sin \phi$, so that the ray path is a circle of radius $c/2\omega$. On the other hand, if ω_0 is treated as constant, integration of (18) again yields the Archimedes spiral $\omega_0 r/c = \phi$.

6. Ehrenfest paradox

Let two circular discs be free to rotate about a common axis. Suppose that the discs are made of a transparent plastic material, so that circular scales on the circumferences can be photographed using an emulsion immediately below the discs. When both discs are stationary markings on the identical scales appear superimposed on a photograph. If, however, one disc rotates relative to an inertial frame, whilst the other remains stationary, what now would an instantaneous photograph reveal? We may assume that the beam of light passing normally through both discs is obtained by collimating (by means of a lens) light from a point source of short duration on the axis (for example, a spark). Lorentz contraction of the circumference would introduce asymmetry without proportionate contraction of the radius. But how can one explain contraction of the radius—the Ehrenfest paradox?

Length contraction in special relativity is a direct consequence of the relativity of simultaneity (as indeed are all relativistic effects). Let two metre sticks have relative velocity v in the direction of their lengths, and let observers S and S′ be at rest relative to the respective sticks. Consider what would be the expectations of S and S′ for a photograph of the superimposed scales, first without contraction of the moving scale and then with contraction. The records on the emulsion of superimposed scale markings some distance apart constitute two events. If the events are simultaneous for S they cannot be simultaneous for S′ due to the constancy of light velocity. In fact the events occur a time τ' apart, according to S′, where

$$\tau' = \frac{L'/2}{c-v} - \frac{L'/2}{c+v} = \frac{vL'/c^2}{1-v^2/c^2} \tag{19}$$

L' being the distance between the events according to S′. Thus, without recognition of the Lorentz contraction, the situation would be that S predicts equal lengths, but S′ predicts length $L' + v\tau'$ as opposed to L' for the moving stick, due to movement of the stick between the photographs. The photographs can reveal only one result, so here is a paradox. In order to resolve the paradox it is necessary for each observer to agree to contract by some factor f the scale which moves relative to him. Then S expects L for his own stick, and L/f for the other stick. S′ expects L' for his own stick and $(L' + v\tau')/f$ for the other stick. There is no contradiction if

$$L/f = L' \qquad (L' + v\tau')/f = L. \tag{20}$$

From (19) and (20) we readily deduce that $f = \gamma$, where $\gamma = (1 - v^2/c^2)^{-1/2}$.

In deciding whether or not to contract the circular scale on the circumference of a rotating disc it is crucial to know whether the superimposed scales can be photographed in such a manner as to permit the stationary and rotating observers to agree on the simultaneity of the recordings of different parts of the scales. This was not possible for uniform linear motion in the direction of the scales, but of course is possible for uniform linear motion normal to the scales. The pulse of light illuminating the scales may come from a spark on the axis above the scales after collimation by a lens. We have seen in § 5 that the observers will differ concerning the shape of light rays proceeding from the axis to the circumference (projecting ray paths onto the plane of the discs). Despite this effect—the curvature of rays for the rotating observer S′—both observers should agree that all portions of the superimposed scales are photographed simultaneously. The light transit time from the axis to a point on the circumference must be independent of the direction of the ray for S′ as well as S; this implies that the wavefronts for S′ are not normal to the rays, as happens for the extraordinary ray in birefringence. Since it is possible to photograph the superimposed scales in a manner which appears simultaneous to both observers, it follows that the photograph will reveal *no* relative contraction of the scales. This is contrary to what is currently believed.

Whilst there will be no relative contraction of the circular scales, in fact both scales will be equally contracted for the rotating observer S′ relative to their size for the inertial observer S. This will be quite clear from the discussion of the clock paradox for circular motion in § 7.

The connection between the Lorentz contraction and the Penrose–Terrell rotation of a two-dimensional object (Penrose 1959, Terrell 1959, McGill 1968) is of interest for rotational motion. Consider, firstly, linear motion of a square in the direction of one side OX, and let an instantaneous photograph be taken using light travelling in the direction of a perpendicular side OY. Due to the finite transit time of light over the distance OY, the point O moves out of the way of a light signal from Y, allowing OY to be seen in projection; the projected length actually photographed is $(v/c)L$ for side L. Due to Lorentz contraction the side OX appears on the photograph as $(1 - v^2/c^2)^{1/2}L$.

Interpreting the latter as a projection, we see that the object appears undistorted on the photograph, but rotated through angle α where $\sin\alpha = v/c$. When we apply this argument to rotational motion of XOY, and when the rotating system is perfectly isolated (i.e. a universe), then the curvature of light rays causes the rotating object to be viewed from a slightly different perspective; the change in the angle of viewing is just sufficient to counteract the Penrose–Terrell rotation (figure 2). The picture obtained now is simply $OX = (1 - v^2/c^2)^{1/2}L$ due to Lorentz contraction (note that light transit times to C are the same from O and from X) together with $OY = (1 - v^2/c^2)^{1/2}L$ due to the reduced phase velocity of light in the direction OY. For a local rotating system the Penrose–Terrell rotation will not be counteracted in this way.

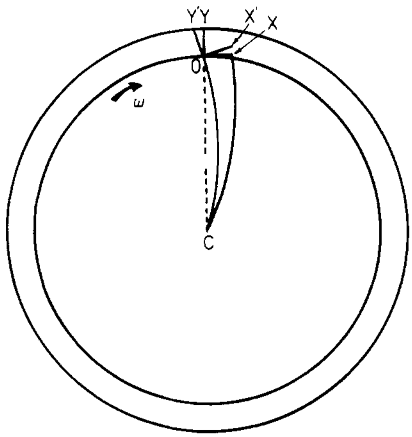

Figure 2. Penrose–Terrell rotation of object XOY rotating at radial distance r and viewed from the centre of rotation C when background matter rotates. $OX = OY = L$ and $OX' = OY' = L(1 - \omega^2 r^2/c^2)^{1/2}$.

7. The clock paradox for circular motion

When accelerations (relative to an inertial frame) are permitted, it is possible for two clocks to start from coincidence and after performing different motions to return to coincidence. At a coincidence a comparison of clock readings is independent of the method of synchronization. The question therefore arises as to whether the clocks will record the same or different times between the two events. If the accelerations of the clocks differ, as will be true in general, then the different time dilation factors associated with the different motions should imply a time difference, but if this is the only effect to consider the observers who remain at rest relative to the respective clocks would differ concerning which goes slower—the clock paradox. The resolution of the clock paradox lies in the recognition of an additional effect, the change of clock rate with change of gravitational potential.

7.1. Linear return motion of one clock

Consider, in the first instance, the case where one clock remains at rest and the other performs linear return motion along the same path. The reversal of the motion at Q is accomplished by uniform deceleration g lasting for time T, where $g \to \infty$ and $T \to 0$ consistently with $gT = 2v$. At the initial event and also at the final event both clocks are coincident at P. Observer S is at rest relative to the clock which remains at P, whilst observer S′ remains at rest relative to the clock which performs the return journey. The distance PQ equals x for S and x' for S′. We may suppose that either observer can monitor the clock remote from him by reception of light pulses emitted at the beats of the clock. Thus each observer may monitor continually the time-keeping of both his own clock and the clock of the other observer.

Firstly, consider the inertial observer S. The number of beats recorded by his own clock between the two events will be simply $(2x/v)\nu_0$. On monitoring the remote clock, he receives pulses at the Doppler-reduced rate ν_- for a time $x/v + x/c$ (since pulses en route continue to arrive for a time x/c after the turn-around point Q is reached) and at the Doppler-increased rate ν_+ for a time $x/v - x/c$. Noting that $\nu_\pm = (1 \pm v/c)\gamma\nu_0$ we may summarize the findings of S by:

$$
\begin{aligned}
N(\mathrm{C}) &= 2x\nu_0/v \\
N(\mathrm{C}') &= \left(\frac{x}{v} + \frac{x}{c}\right)\nu_- + \left(\frac{x}{v} - \frac{x}{c}\right)\nu_+ = \frac{2x\nu_0}{\gamma v}.
\end{aligned}
\tag{21}
$$

Consider, now, the co-moving observer S′. For him the distance travelled is x' rather than x, so he counts $(2x'/v)\nu_0$ beats by his own clock. Monitoring the remote clock, S′ finds exactly the same effects as did S—that is, the Doppler and light transit time effects, which combine to yield simply time dilation. But there is one very important additional effect. During the period T of deceleration there arises for S′ a cosmological gravitational field (in accordance with the principle of equivalence). This means that his clock and the remote clock are at different gravitational potentials, and hence go at different rates. The difference of potential is gx, and the extra number of counts from the remote clock during T is just $(gx/c^2)\nu_0 T$. Using $gT = 2v$, this correction becomes $2vx\nu_0/c^2$, which remains finite as $T \to 0$. In the correction term we have used x rather than x' because the relative velocity of S and S′ is zero in the limit $T \to 0$. Summarizing, S′ will make the counts:

$$
\begin{aligned}
N'(\mathrm{C}') &= 2x'\nu_0/v = 2x\nu_0/\gamma v \\
N'(\mathrm{C}) &= \left(\frac{x'}{v} + \frac{x'}{c}\right)\nu_- + \left(\frac{x'}{v} - \frac{x'}{c}\right)\nu_+ + \frac{2vx\nu_0}{c^2} = \frac{2x\nu_0}{v}.
\end{aligned}
\tag{22}
$$

From (21) and (22) it is clear that both observers can agree that the clock of S′ (the accelerated clock) performs less beats between the two coincidences. The agreement between the observers is achieved only by inclusion of the gravitational frequency change.

The gravitational effect occurs only during the period T of deceleration. It may be eliminated by introducing a third clock C″ at rest relative to observer S″ who completes the return leg of the journey with uniform velocity v. At the instant when C′ passes Q the clock C″ also is passing Q, and an instantaneous photograph of the superimposed clocks permits a comparison of readings. In order to understand how the paradox is now resolved, let two clocks be introduced at rest relative to S, $\mathrm{C_P}$ at P and $\mathrm{C_Q}$ at Q. For S these two clocks are synchronous, but in order that S′ should consider them synchronous $\mathrm{C_Q}$ must be advanced relative to $\mathrm{C_P}$ by vx/c^2, and in order that S″ should consider them synchronous C_P must be advanced relative to C_Q by the same amount vx/c^2 (as follows from the Lorentz transformation). The $2vx\nu_0/c^2$ term of (22) again emerges, resolving the paradox (Lowry 1963). It seems that the gravitational frequency change basically is a variation with time of the synchronization correction applied when the relative motion is uniform (Levi 1967).

Another way to view the effect is to consider the procedure by which C_P and C_Q are synchronized by S, during the time T of deceleration. Due to the deceleration the relative velocity between source and detector alters by gx/c during the light transit time x/c. Hence there arises during T a Doppler shift of vgx/c^2. Using $gT=2v$, the extra beats counted over T amount to precisely $2vx\nu_0/c^2$.

7.2. Circular return motion of one clock

Now let clock C′ move with uniform circular velocity v once around a circular orbit of radius r. During the motion S receives pulses from the clock C′ of S′, which we shall suppose to be relayed via an oscillator at the centre O of the circle. According to S′ the clock C of S will rotate, along with the rest of the universe, about O.

The transverse Doppler shift takes the place of the combined first-order Doppler shifts and the light transit time effect which together amount to time dilation. Thus the effects which led to (21) still are present for circular motion in the view of S.

The additional effects which S′ took into account for linear motion were (i) Lorentz contraction of the distance travelled, and (ii) the gravitational frequency shift. We shall assume that the circumference of the circle appears contracted to S′. But how can one obtain a gravitational frequency shift since C and C′ remain at the same gravitational potential throughout the circular motion (although the centre O of the circle is at a higher gravitational potential)? The gravitational effect which now resolves the paradox is rather different—in fact the curvature of the light rays in the frame of S′. Due to this curvature signals from C are emitted not normally to the motion of C, but at a slightly forward angle which introduces a small first-order Doppler effect. From the aberration formula one find that the angle of emission relative to the forward direction of motion is θ, where $\cos\theta = v/c = \beta$. The total Doppler effect therefore becomes

$$\frac{\nu}{\nu_0}=\frac{(1-\beta^2)^{1/2}}{1-\beta\cos\theta}=\frac{1}{(1-\beta^2)^{1/2}}=\gamma. \tag{23}$$

Thus the small first-order Doppler effect reverses the transverse Doppler effect, which is exactly what is required to resolve the paradox.

The agreement of the observers concerning the difference of time recorded by C and by C′ means that this is a real physical effect. It means that S and S′ will measure different angular velocities of rotation, which are ω and ω_0 respectively as given by (8). Thus the difference is a consequence of the Thomas precession $\omega-\omega_0$.

8. Frequency of signals from rotating sources

The effects which were required to resolve the clock paradox for circular motion are capable of being directly observed. If one monitors at the centre of rotation signals from a source which rotates relative to an inertial frame, then only the transverse Doppler shift (or time dilation) will be observed. The Mössbauer effect has been used to verify this (Hay *et al* 1960, Kundig 1963, Champaney *et al* 1965). In a co-rotating frame of reference there is no transverse Doppler shift, but instead there arises a gravitational frequency shift of the same magnitude due to a cosmological field.

The above are two limiting cases. What happens if we view a rotating source from a frame of reference with different angular velocity? That is, the source continues to rotate with respect to the non-inertial frame. In this situation the small first-order Doppler effect due to curvature of light rays will be seen—precisely the effect taken into account in (23). If two such sources rotate in opposite directions with the same velocity, then since the first-order Doppler shift will be in the one case a red shift and in the other a blue shift (the curvature of the rays being the same for each). The difference in the clock frequencies will therefore be given by

$$\frac{\Delta\nu}{\nu_0}=\frac{(1-\beta^2)^{1/2}}{1-\beta\cos\theta}-\frac{(1-\beta^2)^{1/2}}{1+\beta\cos\theta}=\frac{(1-\beta^2)^{1/2}2\beta\cos\theta}{1-\beta^2\cos^2\theta} \tag{24}$$

where $\pm\beta c(=\pm v)$ are the velocities of the clocks with respect to the rotating frame, and where $\cos\theta=\Omega r/c$, where Ω is the angular velocity of the rotating frame relative to the inertial frame. Between successive coincidences of the clocks, a time of $2\pi r/v$, the difference in the number of beats for the two clocks will be ΔN where

$$\Delta N=(2\pi rv^{-1})\Delta\nu\simeq(2\pi rv^{-1})(2\beta\cos\theta\,\nu_0)=4A\Omega\nu_0c^{-2}. \tag{25}$$

Here A $(=\pi r^2)$ is the area enclosed by the path of the clocks.

Hafele and Keating (1972) (see also Schlegel 1974) have performed such an experiment. ^{133}Cs clocks, operating on a hyperfine transition of frequency $9{\cdot}192631770\times10^9$ Hz, were flown in opposite directions once around the Earth. The difference in their recorded times over this journey was 332 ± 17 ns, which agrees with the predicted time difference of 315 ns.

9. The Sagnac effect

Let a plane light wave be split by a partial reflector into two waves which then propagate in opposite senses around a closed path and are finally re-united at the partial reflector. When the entire optical system is rotated relative to an inertial frame, Sagnac (1915) discovered that the relative phase of the two beams alters by an amount given by (25). If the mirror system is used as the cavity of a ring laser, the phase difference between the clockwise and counterclockwise propagating waves introduces a frequency difference given by (24) (for a square path of side D this reduces to $\Omega D/c$) between modes which would be degenerate if the system were at rest inertially. The frequency difference between the modes is readily measured in the output of a detector into which is fed emissions in the two laser modes (Macek and Davis 1963, Cheo and Heer 1964).

The explanation of the Sagnac effect is simple for the inertial frame of reference. The motion of the mirrors during the light transit time between mirrors causes the clockwise and counterclockwise waves to be reflected at different points of space, which leads to an optical path difference.

With respect to the co-rotating frame the mirrors are at rest, but again there arises an optical path difference due to the different curvatures of the clockwise and counterclockwise waves. Alternatively, one may compare phase velocities around one and the same path. If $\boldsymbol{v}(r)$ is the aether rotation field, the difference of phase velocities of the two waves around a closed path will produce the phase change,

$$\Delta N = 2b\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = 2b\int\int \nabla\times\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = 4b\int\int \boldsymbol{\omega}_0 \cdot \mathrm{d}\boldsymbol{S} \tag{26}$$

where $b = \nu_0/c^2$.

The Sagnac effect has been thoroughly reviewed by Post (1967). Post considers the effect of a medium of refractive index n which may or may not move with the mirror system. Defining a drag coefficient by $\alpha = 1 - n^{-2} - \delta(\ln n)/\delta(\ln \lambda)$ Post obtains the following results for the Sagnac phase change:

interferometer stationary medium stationary	$\Delta N = 0$	
interferometer stationary medium rotating	$\Delta N = 2b\oint n^2\alpha\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$	
interferometer rotating medium stationary	$\Delta N = 2b\oint n^2\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$	(27)
interferometer rotating medium rotating	$\Delta N = 2b\oint n^2(1-\alpha)\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$.	

Post treats Sagnac effects both from geometrical optics and physical optics. For the latter treatment the refractive index of the aether is required. Post obtains this, or rather the constitutive tensor, from a metric obtained from the Minkowski metric by a Galilean rotation.

A point that has not been made elsewhere is that the Sagnac experiment must give a null result (like aether drift experiments of the Michelson–Morley and Kennedy–Thorndike type) when the rotating system is an entire universe. Otherwise the angular velocity of the universe could be measured absolutely, or at least, relative to some standard external to the universe, and this is impossible if the universe is a perfectly isolated system. But how does the Sagnac phase change disappear in these circumstances?

When the rotating system is local, the Sagnac phase change is attributed, in the inertial frame, to a difference of optical path due to reflections at different points of space for the clockwise and counterclockwise propagating waves, and is attributed, in the co-rotating frame, to a difference of optical path due to the different curvatures of the clockwise and counterclockwise rays. If these two effects were observed in the same frame of reference they would cancel each other; the first effect is due to rotation of the system, and the second effect is due to rotation of background matter, so that the Sagnac effect does indeed disappear when both system and background rotate. Clearly there will be no effect when both system and background are stationary. The disappearance of the Sagnac phase change for a rotating universe is a good example of the fundamental principles discussed in § 3.

10. Gravitational fields in rotating frames of reference

The adoption, by convention, of a flat space–time, with its implication of non-holonomic aether motion (§ 2), requires the introduction of a Newtonian-type gravitational field. Newton's gravitational field is obtained from a scalar potential, but for rotating isolated systems, or equivalently, for rotating frames of reference, a vector potential must also be introduced.

In a rotating frame of reference there must arise a gravitational field which replaces the centrifugal force on the Syncom satellite in an inertial frame (the Syncom satellite remains permanently overhead since its angular velocity with respect to an inertial frame equals the spin angular velocity of the Earth). This is an outward-directed force. There must arise also a gravitational force, directed inwards, which balances the centrifugal force on a distant star with respect to a rotating frame. In order to provide for the stationary Syncom satellite an outward-directed force and for the rotating star an inward-directed force, we shall see that a magnetic-type gravitational field is required.

The required cosmological gravitational fields can be obtained from the scalar and vector potentials (ϕ, $\boldsymbol{A}$), where

$$\begin{aligned}\phi &= \frac{\phi_0}{(1-\omega^2 r^2/c^2)^{1/2}} = (1+\omega_0^2 r^2/c^2)^{1/2}\phi_0 \\ \boldsymbol{A} &= -\frac{\boldsymbol{\omega}\times\boldsymbol{r}\phi_0}{c(1-\omega^2 r^2/c^2)^{1/2}} = \frac{\boldsymbol{\omega}_0\times\boldsymbol{r}\phi_0}{c}.\end{aligned} \tag{28}$$

Here use has been made of relations (8). The potential ϕ_0 is a universal constant with the significance that it gives to any mass a gravitational potential energy equal to minus its rest energy. If we treat the imaginary quantity $\mathrm{i}G^{1/2}M$, where G is the gravitational constant and M is rest mass, analogously to charge in electromagnetism, then ϕ_0 is defined by

$$-\mathrm{i}G^{1/2}M\phi_0 = KMc^2. \tag{29}$$

K is a constant which can be given the magnitude unity by choice of units, but which is not dimensionless; KM is inertial mass if M is gravitational mass. Like imaginary charges attract.

We derive gravitational fields from the potentials (28) exactly as in electromagnetism, and indeed it is convenient to employ the conventional electromagnetic notation for the imaginary fields. Treating ω_0, but not ω, as a constant,

$$\boldsymbol{E} = -\delta\boldsymbol{A}/c\,\delta t - \boldsymbol{\nabla}\phi = -\gamma^{-1}\omega_0^2\boldsymbol{r}\phi_0/c^2 = -\gamma\omega^2\boldsymbol{r}\phi_0/c^2 \tag{30a}$$

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = -2\boldsymbol{\omega}_0\phi_0/c = -2\gamma\boldsymbol{\omega}\phi_0/c. \tag{30b}$$

The force on a mass M due to these fields is given by the Lorentz force law,

$$\boldsymbol{F} = \mathrm{i}G^{1/2}M(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}/c). \tag{31}$$

In the case of the Syncom satellite $\boldsymbol{v}=0$, and substitution from (30*a*) and (29) then yields

$$\boldsymbol{F} = \mathrm{i}G^{1/2}M\boldsymbol{E} = KM\gamma\omega^2\boldsymbol{r} \tag{32}$$

which is what is required to replace relativistic centrifugal force.

In the case of the distant star we have $\boldsymbol{v} = -\boldsymbol{\omega}\times\boldsymbol{r}$, so that

$$\boldsymbol{F} = \mathrm{i}G^{1/2}M[\boldsymbol{E}-(\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{B}/c] = -KM\gamma\omega^2\boldsymbol{r}. \tag{33}$$

Thus the magnetic-type gravitational field (30*b*) yields a Coriolis force which is oppositely directed and twice as great as the gravostatic force on M. Again this is exactly what is required to maintain the circular orbit of the distant star.

It is of interest to note that the Sagnac phase change ΔN, as given by (26), can be expressed in the form,

$$\Delta N = 2\nu_0(\phi_0 c)^{-1}\int\int \boldsymbol{B}\,.\,\mathrm{d}\boldsymbol{S} = 2\nu_0(\phi_0 c)^{-1}\oint \boldsymbol{A}\,.\,\mathrm{d}\boldsymbol{r}. \tag{34}$$

Its analogy to the phase shift of de Broglie waves in the Aharanov–Bohm effect (Erlichson 1970) is now apparent, and also to the phase shift of electrons in a superconductor (Jaklevic *et al* 1965).

Appendix. Metric for a rotating frame of reference

Adopting an inertial frame of reference, and neglecting cosmological terms, the metric for space–time according to Einstein's theory, will be Minkowskian. For cylindrical polar coordinates, r, ϕ, z,

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}r^2 - r^2\mathrm{d}\phi^2 - \mathrm{d}z^2. \tag{A.1}$$

Transforming, now, to a rotating frame by use of

$$\mathrm{d}\phi' = \mathrm{d}\phi - \omega\,\mathrm{d}t \tag{A.2}$$

we obtain

$$\mathrm{d}s^2 = \gamma^{-2}c^2\,\mathrm{d}t^2 - 2r^2\omega\,\mathrm{d}\phi'\,\mathrm{d}t - \mathrm{d}r^2 - r^2\,\mathrm{d}\phi'^2 - \mathrm{d}z^2 \tag{A.3}$$

where $\gamma = (1-\omega^2r^2/c^2)^{-1/2}$.

The space–time (A.3) will be flat like (A.1) if (A.2) is holonomic, which implies that ω is not a function of r in accordance with (8). Whilst space–time would be flat, space itself is not flat. If ω is a function of r due to the Thomas precession, then neither space–time nor space is flat for a rotating frame of reference. Post (1967) considers, in addition to (A.2), the non-holonomic time transformation, $\mathrm{d}t' = \mathrm{d}t/\gamma$. This implies that $\omega\,\mathrm{d}t = \omega_0\,\mathrm{d}t'$, so that (A.2) now can be integrated, but not the equation defining $\mathrm{d}t'$.

Møller (1972) has used metrics (A.1) and (A.3) to resolve the clock paradox for circular motion. Inertial observer S, when monitoring his own clock, substitutes into (A.1) $\mathrm{d}r = \mathrm{d}\phi = \mathrm{d}z = 0$, finding $\mathrm{d}s = c\,\mathrm{d}t$; when monitoring the moving clock, he substitutes $\mathrm{d}\phi = \omega\,\mathrm{d}t$ and $\mathrm{d}r = \mathrm{d}z = 0$, which yields $\mathrm{d}s = \gamma^{-1}c\,\mathrm{d}t$. The non-inertial observer S′, when monitoring his own clock, substitutes into (A.3) $\mathrm{d}r = \mathrm{d}\phi' = \mathrm{d}z = 0$, obtaining $\mathrm{d}s = \gamma^{-1}c\,\mathrm{d}t$; when monitoring the clock moving with respect to his frame, he substitutes $\mathrm{d}\phi' = -\omega\,\mathrm{d}t$ and $\mathrm{d}r = \mathrm{d}z = 0$, obtaining $\mathrm{d}s = c\,\mathrm{d}t$. Both observers will therefore agree that the non-inertial clock goes slow relative to the clock which remains at rest in the inertial frame.

An objection to the metric form (A.3) is that simultaneity has been defined in the inertial frame rather than the rotating frame. If one uses the Einstein definition of simultaneity for a local infinitesimal region, then following Adler *et al* (1965) we would introduce a new time coordinate by the non-holonomic relation,

$$\mathrm{d}t' = \mathrm{d}t - \mathrm{d}\tau \tag{A.4}$$

where

$$\mathrm{d}\tau = \frac{\frac{1}{2}r\,\mathrm{d}\phi'}{c-\omega r} - \frac{\frac{1}{2}r\,\mathrm{d}\phi'}{c+\omega r} = \frac{\omega r^2\,\mathrm{d}\phi'}{c^2-\omega^2r^2}. \tag{A.5}$$

$d\tau$ is just the synchronization correction applied by S to two events with spatial separation $r\,d\phi'$ if the events are synchronous for S'. Substitution of (A.4) and (A.5) into (A.3) yields

$$ds^2 = \gamma^{-2}c^2\,dt'^2 - dr^2 - \gamma^2 r^2\,d\phi'^2 - dz^2. \tag{A.6}$$

The further non-holonomic transformation,

$$d\bar{t} = \gamma^{-1}\,dt' \qquad d\bar{\phi} = \gamma\,d\phi \tag{A.7}$$

will change the metric (A.6) back to the form (A.1).

We may decompose the Lorentz transformation in a similar fashion into a Galilean transformation, a re-synchronization of clocks, and finally a time dilation and length contraction. Thus, the metric,

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{A.8}$$

under the Galilean transformation,

$$x' = x - vt \tag{A.9}$$

becomes

$$ds^2 = (1 - v^2/c^2)c^2\,dt^2 - 2\,dx'v\,dt - dx'^2 - dy^2 - dz^2. \tag{A.10}$$

Re-synchronizing clocks by

$$dt' = dt - \gamma^2 v\,dx'/c^2 \tag{A.11}$$

one obtains

$$ds^2 = \gamma^{-2}c^2\,dt'^2 - \gamma^2\,dx'^2 - dy^2 - dz^2 \tag{A.12}$$

and finally the time dilation and length contraction non-holonomic transformation,

$$d\bar{t} = \gamma^{-1}\,dt' \qquad d\bar{x} = \gamma\,dx' \tag{A.13}$$

leads back again to the metric form (A.8).

Often (e.g. Møller 1972) ω in (A.2) is treated as constant. Then the space–time (A.3) will be flat, although three-space is not flat. The geodesics on this three-space have been considered in some detail by Arzelies (1966), who then goes on to consider different measures of time on the rotating disc. In general authors have not distinguished between a rotating local system and a rotating universe.

The dependence of ω on r given by relations (8) is not the only one that has been proposed. Trocheris (1949) has given grounds for $\omega r/c = \tanh(\omega_0 r/c)$, and Takeno (1952) also has obtained this result. On the other hand, Hill (1946) finds $\omega r = -icJ_1(2i\omega_0 r/c)/J_0(2i\omega_0 r/c)$ where J_1 and J_0 are Bessel functions.

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